

On involutions in extremal self-dual codes and the dual distance of semi self-dual codes.

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Abstract

A classical result of Conway and Pless is that a natural projection of the fixed code of an automorphism of odd prime order of a self-dual binary linear code is self-dual [13]. In this paper we prove that the same holds for involutions under some (quite strong) conditions on the codes.

In order to prove it, we introduce a new family of binary codes: the semi self-dual codes. A binary self-orthogonal code is called semi self-dual if it contains the all-ones vector and is of codimension 2 in its dual code. We prove upper bounds on the dual distance of semi self-dual codes.

As an application we get the following: let \mathcal{C} be an extremal self-dual binary linear code of length $24m$ and $\sigma \in \text{Aut}(\mathcal{C})$ be a fixed point free automorphism of order 2. If m is odd or if $m = 2k$ with $\binom{5k-1}{k-1}$ odd then \mathcal{C} is a free $\mathbb{F}_2\langle\sigma\rangle$ -module. This result has quite strong consequences on the structure of the automorphism group of such codes.

Keywords: semi self-dual codes, bounds on minimum distance, automorphism group, free modules, extremal codes

1. Introduction

The research in this paper is motivated by the study of involutions of extremal self-dual codes, which plays a fundamental role in [18, 6, 5, 8, 7, 22].

Let $m \in \mathbb{N}$ and $\mathcal{C} = \mathcal{C}^\perp \leq \mathbb{F}_2^{24m}$ be an extremal binary self-dual code, so $d(\mathcal{C}) = 4m + 4$ [16]. Then \mathcal{C} is doubly even [20]. There are unique extremal self-dual codes of length 24 and 48 and these are the only known extremal codes of length $24m$. It is an intensively studied open question raised in [21], whether an extremal code of length 72 exists. A series of many papers has shown that if such a code exists, then its automorphism group $\text{Aut}(\mathcal{C}) = \{\sigma \in S_{24m} \mid \sigma(\mathcal{C}) = \mathcal{C}\}$ has order ≤ 5 (see [4] for an exposition of this result). Stefka Bouyuklieva [9] studies automorphisms of order 2 of such codes. She shows that if \mathcal{C} is an extremal code of length $24m$, $m \geq 2$ and $\sigma \in \text{Aut}(\mathcal{C})$ has order 2, then the permutation σ has no fixed points, with one exception, $m = 5$, where there might be 24 fixed points. If $\sigma = (1, 2) \dots, (24m - 1, 24m)$ is a fixed point free automorphism of a doubly even self dual code \mathcal{C} , then its *fixed code*

$$\mathcal{C}(\sigma) := \{c \in \mathcal{C} \mid \sigma(c) = c\}$$

is isomorphic to

$$\pi(\mathcal{C}(\sigma)) = \{(c_1, \dots, c_{12m}) \in \mathbb{F}_2^{12m} \mid (c_1, c_1, c_2, c_2, \dots, c_{12m}, c_{12m}) \in \mathcal{C}\}$$

such that

$$\pi(\{c + \sigma(c) \mid c \in \mathcal{C}\}) = \pi(\mathcal{C}(\sigma))^\perp \subseteq \pi(\mathcal{C}(\sigma)).$$

As \mathcal{C} is doubly-even, all words in $\pi(\mathcal{C}(\sigma))$ have even weight. It is shown in [18] and [5] that the code \mathcal{C} is a free $\mathbb{F}_2\langle\sigma\rangle$ -module, if and only if $\pi(\mathcal{C}(\sigma))$ is self-dual. If $\pi(\mathcal{C}(\sigma))$ is not self-dual then it contains the dual \mathcal{D}^\perp of some code \mathcal{D} of length $12m$ with

$$\mathbf{1} := (1, \dots, 1) \in \pi(\mathcal{C}(\sigma))^\perp \subseteq \mathcal{D} \subseteq \mathcal{D}^\perp \subseteq \pi(\mathcal{C}(\sigma)).$$

In particular $d(\mathcal{D}^\perp) \geq d(\pi(\mathcal{C}(\sigma))) = \frac{1}{2}d(\mathcal{C}(\sigma)) \geq \frac{1}{2}d(\mathcal{C})$.

Definition 1.1. A binary self-orthogonal code $\mathcal{D} \subseteq \mathcal{D}^\perp \leq \mathbb{F}_2^n$ of length n is called *semi self-dual*, if $\mathbf{1} := (1, \dots, 1) \in \mathcal{D}$ and $\dim(\mathcal{D}^\perp/\mathcal{D}) = 2$.

Self-orthogonal codes always consist of words of even weight, so $\text{wt}(c) := |\{i \mid c_i = 1\}| \in 2\mathbb{Z}$ for all $c \in \mathcal{D}$. Hence already the condition that $\mathbf{1} \in \mathcal{D}$

implies that the length $n = 12m$ of \mathcal{D} is even. Note that $\mathcal{D}^\perp \subseteq \mathbf{1}^\perp = \{c \in \mathbb{F}_2^n \mid \text{wt}(c) \in 2\mathbb{Z}\}$ implies that also \mathcal{D}^\perp consists of even weight vectors. The *dual distance* of \mathcal{D} is the minimum weight of the dual code $\text{dd}(\mathcal{D}) := d(\mathcal{D}^\perp) := \min(\text{wt}(\mathcal{D}^\perp \setminus \{0\}))$.

In this paper we will bound the dual distance $\text{dd}(\mathcal{D}) = d(\mathcal{D}^\perp)$ of semi self-dual codes. In particular if the length of \mathcal{D} is $12m$ with either m odd or $m = 2\mu$ such that $\binom{5\mu-1}{\mu-1}$ is odd, then $\text{dd}(\mathcal{D}) \leq 2m$ (see Theorem 2.1 below for the general statement).

Then we may conclude the following Theorem.

Theorem 1.2. *Let $\mathcal{C} = \mathcal{C}^\perp \leq \mathbb{F}_2^{24m}$ be an extremal code of length $24m$ and $\sigma \in \text{Aut}(\mathcal{C})$ be a fixed point free automorphism of order 2. Then \mathcal{C} is a free $\mathbb{F}_2\langle\sigma\rangle$ -module if m is odd or if $m = 2\mu$ with $\binom{5\mu-1}{\mu-1}$ odd.*

In particular, for $m = 3$, we obtain [18, Theorem 3.1] without appealing to the classification of all extremal codes of length 36 in [1] and without any serious computer calculation.

Remark 1.3. *In [23], Zhang proved that extremal self-dual binary linear codes of length a multiple of 24 may exist only up to length $3672 = 153 \cdot 24$. About 72% of these lengths are covered by Theorem 1.2. In particular the projections of fixed codes by fixed point free involutions in self-dual [96, 48, 20] and [120, 60, 24] codes (see [11, 10] for an exposition of the state of the art for the codes with these parameters) are self-dual.*

The same arguments as in [18] can now be applied to obtain the following quite strong consequence on the structure of the automorphism group of such extremal codes.

Corollary 1.4. *Let $m \geq 3$ be odd and assume that $m \neq 5$. Let $\mathcal{C} = \mathcal{C}^\perp \leq \mathbb{F}_2^{24m}$ be an extremal code. If 8 divides $|\text{Aut}(\mathcal{C})|$ then a Sylow 2-subgroup of $\text{Aut}(\mathcal{C})$ is isomorphic to $C_2 \times C_2 \times C_2$, $C_2 \times C_4$ or D_8 .*

Proof. Let S be a Sylow-2-subgroup of $\text{Aut}(\mathcal{C})$.

By our assumption and [9] all elements of order 2 in $\text{Aut}(\mathcal{C})$ act without fixed points on the places $\{1, \dots, 24m\}$. This immediately implies that all S -orbits have length $|S|$, so $|S|$ divides $24m$ and hence $|S| = 8$.

So we only need to exclude $S = C_8$ and $S = Q_8$. This is done by considering the module structure of \mathcal{C} as an $\mathbb{F}_2 S$ -module. Note that both groups have a unique elementary abelian subgroup, say Z , and $Z \cong C_2$. By Theorem 1.2

the module \mathcal{C} is a free $\mathbb{F}_2 Z$ -module. Chouinard's Theorem [12] states that a module is projective if and only if its restriction to every elementary abelian subgroup is projective. Then \mathcal{C} is also a free $\mathbb{F}_2 S$ -module of rank

$$\text{rk}_{\mathbb{F}_2 S}(\mathcal{C}) = \frac{\dim_{\mathbb{F}_2}(\mathcal{C})}{|S|} = \frac{12m}{8} = 3 \cdot \frac{m}{2} \notin \mathbb{N}$$

a contradiction. □

Remark 1.5. *Note that the cyclic group C_8 is already excluded by the Sloane-Thompson Theorem (see also [15]) because $S \cong C_8$ acting fixed point freely on $24m$ points implies that S is not in the alternating group, so S does not fix any doubly-even self-dual code.*

2. Bounds on the dual distance of semi self-dual codes

In the previous section we introduced the definition of semi self-dual codes. Now we will prove upper bounds on their dual distance. Even if this family of codes was introduced as a tool for the proof of Theorem 1.2, it seems to be interesting also by itself. Applying the methods from [20], we show the following theorem.

Theorem 2.1. *Let $\mathcal{D} \leq \mathbb{F}_2^n$ be a semi self-dual code. Then the dual distance of \mathcal{D} is bounded by*

$$\text{dd}(\mathcal{D}) = d(\mathcal{D}^\perp) \leq \begin{cases} 4\lfloor \frac{n}{24} \rfloor + 2 & \text{if } n \equiv 0, 2, 4, 6, 8, 10, 12, 14 \pmod{24} \\ 4\lfloor \frac{n}{24} \rfloor + 4 & \text{if } n \equiv 16, 18, 20 \pmod{24} \\ 4\lfloor \frac{n}{24} \rfloor + 6 & \text{if } n \equiv 22 \pmod{24}. \end{cases}$$

If $n = 24\mu$ for some integer μ and \mathcal{D} is doubly-even or $\binom{5\mu-1}{\mu-1}$ is odd then

$$\text{dd}(\mathcal{D}) = d(\mathcal{D}^\perp) \leq 4\mu.$$

Theorem 2.1 follows by combining Remark 3.1, Proposition 4.1, Proposition 5.2 and Proposition 5.3.

Remark 2.2. *The well-known Kummer's theorem on binomial coefficients implies that $\binom{5\mu-1}{\mu-1}$ is odd if and only if there are no carries when 4μ is added to $\mu - 1$ in base 2.*

By direct calculations with MAGMA, using a database [17] of all self-dual binary linear codes of length up to 40, most of the bounds of Theorem 2.1 can be shown to be sharp. In particular, we have semi self-dual codes such that their dual codes have parameters $[4, 3, 2]$, $[6, 4, 2]$, $[8, 5, 2]$, $[10, 6, 2]$, $[12, 7, 2]$, $[14, 8, 2]$, $[16, 9, 4]$, $[18, 10, 4]$, $[20, 11, 4]$ and $[22, 12, 6]$ and a doubly-even semi self-dual code with dual code of parameters $[24, 13, 4]$.

3. Self-dual subcodes

From now on let \mathcal{D} be a semi self-dual code of even length $n \geq 4$. Furthermore, let $\mu = \lfloor \frac{n}{24} \rfloor$.

Remark 3.1. *There are exactly three self-dual codes $\mathcal{C}_i = \mathcal{C}_i^\perp$ ($i \in \{1, 2, 3\}$) with*

$$\mathcal{D} \subset \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset \mathcal{D}^\perp.$$

From the bound on $d(\mathcal{C}_i)$ given in [20, Theorem 5] we obtain

$$dd(\mathcal{D}) = d(\mathcal{D}^\perp) \leq d(\mathcal{C}_1) \leq \begin{cases} 4\mu + 6 & \text{if } n \equiv 22 \pmod{24} \\ 4\mu + 4 & \text{otherwise.} \end{cases}$$

We aim to find a better bound.

4. Shadows: the doubly-even case

Proposition 4.1. *If \mathcal{D} is doubly-even, then*

$$d(\mathcal{D}^\perp) \leq \begin{cases} 4\mu & \text{if } n \equiv 0 \pmod{24} \\ 4\mu + 2 & \text{if } n \equiv 4, 8, 12 \pmod{24} \\ 4\mu + 4 & \text{if } n \equiv 16, 20 \pmod{24}. \end{cases}$$

Proof. Since every doubly-even binary linear code is self-orthogonal, \mathcal{D}^\perp cannot be doubly-even and so in \mathcal{D}^\perp there exists a codeword of weight $w \equiv 2 \pmod{4}$. Thus we can take $\mathcal{D} < \mathcal{F} = \mathcal{F}^\perp < \mathcal{D}^\perp$ with \mathcal{F} not doubly-even, so that $\mathcal{D} = \mathcal{F}_0 := \{f \in \mathcal{F} \mid \text{wt}(f) \equiv 0 \pmod{4}\}$ is the maximal doubly-even subcode of \mathcal{F} .

Let $S(\mathcal{F}) := \mathcal{D}^\perp - \mathcal{F}$ denote the shadow of \mathcal{F} . By [3],

$$2d(\mathcal{F}) + d(S(\mathcal{F})) \leq 4 + \frac{n}{2}. \tag{1}$$

Note that $d(\mathcal{D}^\perp) = \min\{d(\mathcal{F}), d(S(\mathcal{F}))\}$, since $\mathcal{D}^\perp = S(\mathcal{F}) \cup \mathcal{F}$. Since we have the bound (1), the maximum for $\min\{d(\mathcal{F}), d(S(\mathcal{F}))\}$ is reached if

$$d(\mathcal{D}^\perp) = d(\mathcal{F}) = d(S(\mathcal{F})) = \left\lfloor \frac{4 + \frac{n}{2}}{3} \right\rfloor$$

so that

$$d(\mathcal{D}^\perp) \leq \left\lfloor \frac{8 + n}{6} \right\rfloor,$$

which yields the proposition since $d(\mathcal{D}^\perp)$ is even. \square

In [19] Rains proved more general bounds on the dual distance of doubly-even binary linear codes, without assuming that they contain the all-ones vector.

Length	Rains' bound	Our bound
24μ	$4\mu + 4$	4μ
$24\mu + 4$	$4\mu + 2$	$4\mu + 2$
$24\mu + 8$	$4\mu + 4$	$4\mu + 2$
$24\mu + 12$	$4\mu + 2$	$4\mu + 2$
$24\mu + 16$	$4\mu + 4$	$4\mu + 4$
$24\mu + 20$	$4\mu + 4$	$4\mu + 4$

With our additional assumption there is a substantial improvement in particular for lengths divisible by 24.

5. Weight enumerators: the non doubly-even case.

In this section we assume that \mathcal{D} is not doubly-even. We will use the following notation:

- $N := \frac{n}{2}$, $2d := d(\mathcal{D}^\perp)$;
- $A(x, y) := W_{\mathcal{D}}(x, y) = \sum_{c \in \mathcal{D}} x^{n-\text{wt}(c)} y^{\text{wt}(c)} = x^{2N} + \sum_{i=d}^{N-d} a_i x^{2N-2i} y^{2i} + y^{2N}$ the weight enumerator of \mathcal{D} ;
- $D(x, y) := A\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) = \frac{1}{2}x^{2N} + \sum_{i=d}^{N-d} d_i x^{2N-2i} y^{2i} + \frac{1}{2}y^{2N}$, so that $2D$ is the weight enumerator of \mathcal{D}^\perp ;
- $B(x, y) := A(x, y) - D(x, y) = \frac{1}{2}x^{2N} + \sum_{i=d}^{N-d} b_i x^{2N-2i} y^{2i} + \frac{1}{2}y^{2N}$;

- $F(x, y) := B\left(\frac{x+y}{\sqrt{2}}, i\frac{x-y}{\sqrt{2}}\right) = \frac{1}{2} \left(W_{S(\mathcal{D})}(x, y) - W_{S(\mathcal{D})}\left(\frac{1+i}{\sqrt{2}}x, \frac{1-i}{\sqrt{2}}y\right) \right)$, where $S(\mathcal{D}) = \mathcal{D}_0^\perp - \mathcal{D}^\perp$ is the shadow of \mathcal{D} .

The polynomial $B(x, y)$ is anti-invariant under the MacWilliams transformation $H : (x, y) \mapsto 1/\sqrt{2}(x + y, x - y)$ and invariant under the transformation $I : (x, y) \mapsto (x, -y)$, so by [2, Lemma 3.2]

$$B(x, y) \in (x^4 - 6x^2y^2 + y^4) \cdot \mathbb{C}[x^2 + y^2, x^2y^2(x^2 - y^2)^2].$$

and we can write

$$B(x, y) = (x^4 - 6x^2y^2 + y^4) \cdot \sum_{i=0}^{\lfloor \frac{N-2}{4} \rfloor} e_i (x^2 + y^2)^{N-2-4i} (x^2y^2(x^2 - y^2)^2)^i \quad (2)$$

and, consequently,

$$F(x, y) = 2(x^4 + y^4) \cdot \sum_{i=0}^{\lfloor \frac{N-2}{4} \rfloor} e_i (2xy)^{N-2-4i} \left(-\frac{1}{4}x^8 + \frac{1}{2}x^4y^4 - \frac{1}{4}y^8 \right)^i. \quad (3)$$

Notice that (3) implies that the degrees of the monomials of $F(x, y)$ are congruent to $N - 2 \pmod{4}$. Since

$$\begin{aligned} F(x, y) &= \frac{1}{2} \left(W_{S(\mathcal{D})}(x, y) - W_{S(\mathcal{D})}\left(\frac{1+i}{\sqrt{2}}x, \frac{1-i}{\sqrt{2}}y\right) \right) = \\ &= \frac{1}{2} \left(W_{S(\mathcal{D})}(x, y) - i^N W_{S(\mathcal{D})}(x, -iy) \right), \end{aligned}$$

it is easy to see that $F(x, y)$ is the weight enumerator of the following set

$$\mathcal{S} := \{s \in S(\mathcal{D}) \mid \text{wt}(s) \equiv N - 2 \pmod{4}\}.$$

So the coefficients of $F(x, y)$ are non-negative integers.

Then we get the following.

Corollary 5.1. *Let e_i be as in (2) and (3) and put $\epsilon_i := (-1)^i 2^{N-1-6i} e_i$. Then all ϵ_i are non-negative integers.*

Proof. We have

$$F(1, y) = (1 + y^4)y^{N-2} \cdot \sum_{i=0}^{\lfloor \frac{N-2}{4} \rfloor} \epsilon_i y^{-4i} (1 - y^4)^{2i}.$$

with $\epsilon_i := (-1)^i 2^{N-1-6i} e_i$. Substitute $\lfloor \frac{N-2}{4} \rfloor - i = h$.

$$F(1, y) = y^{N-2-4\lfloor \frac{N-2}{4} \rfloor} (1 + y^4)(1 - y^4)^{2\lfloor \frac{N-2}{4} \rfloor} \cdot \sum_{h=0}^{\lfloor \frac{N-2}{4} \rfloor} \epsilon_{\lfloor \frac{N-2}{4} \rfloor - h} (y^4(1 - y^4)^{-2})^h.$$

Let $r := N - 2 - 4\lfloor \frac{N-2}{4} \rfloor$. Note that r is the remainder of the division of $N - 2$ by 4.

$$\begin{aligned} F(1, y) &= \sum_{j=0}^{2N} f_j y^j = f_0 + \dots + f_{r-1} y^{r-1} + y^r \sum_{j=r}^{2N} f_j y^{j-r} \\ &= y^r (1 + y^4)(1 - y^4)^{2\lfloor \frac{N-2}{4} \rfloor} \cdot \sum_{h=0}^{\lfloor \frac{N-2}{4} \rfloor} \epsilon_{\lfloor \frac{N-2}{4} \rfloor - h} (y^4(1 - y^4)^{-2})^h. \end{aligned}$$

Then $f_j = 0$ if $j \not\equiv r \pmod{4}$. Set $Z = y^4$. Then

$$\sum_k f_{4k+r} Z^k = (1 + Z)(1 - Z)^{2\lfloor \frac{N-2}{4} \rfloor} \cdot \sum_{h=0}^{\lfloor \frac{N-2}{4} \rfloor} \epsilon_{\lfloor \frac{N-2}{4} \rfloor - h} (Z(1 - Z)^{-2})^h.$$

Put

$$f(Z) := (1 + Z)^{-1} (1 - Z)^{-2\lfloor \frac{N-2}{4} \rfloor}, \quad g(Z) := Z(1 - Z)^{-2}.$$

Then there are coefficients $\gamma_{h,k}$ such that

$$Z^k f(Z) = \sum_{h=0}^{\lfloor \frac{N-2}{4} \rfloor} \gamma_{h,k} g(Z)^h.$$

Since $g(0) = 0$ and $g'(0) \neq 0$, we can apply the Bürmann-Lagrange theorem (see [20, Lemma 8]) to obtain

$$\gamma_{h,k} = [\text{coeff. of } Z^{h-k} \text{ in } (1 - Z)^{-1-2\lfloor \frac{N-2}{4} \rfloor+2h}] = \binom{2\lfloor \frac{N-2}{4} \rfloor - h - k}{h - k} > 0.$$

In particular

$$\epsilon_{\lfloor \frac{N-2}{4} \rfloor - h} = \sum_{k=0}^{\lfloor \frac{h-r}{4} \rfloor} \gamma_{h,k} f_{4k+r}$$

is a non-negative integer for all h . □

Proposition 5.2. *If \mathcal{D} is not doubly-even and $n \equiv 0, 2, 4, 6, 8, 10, 12, 14 \pmod{24}$ then $d(\mathcal{D}^\perp) \leq 4\mu + 2$.*

Proof. We have that

$$\begin{aligned} B(1, Y) &= 1/2 + \sum_{j=d}^{N-d} b_j Y^j + 1/2 Y^N \\ &= (1 - 6Y + Y^2)(1 + Y)^{N-2} \cdot \sum_{i=0}^{\lfloor \frac{N-2}{4} \rfloor} e_i (Y(1 - Y)^2(1 + Y)^{-4})^i. \end{aligned}$$

Let

$$f(Y) := (1 - 6Y + Y^2)^{-1}(1 + Y)^{2-N}, \quad g(Y) := Y(1 - Y)^2(1 + Y)^{-4}.$$

As before we find coefficients $\alpha_i(N)$ such that

$$f(Y) = \sum_{i=0}^{\lfloor \frac{N-2}{4} \rfloor} \alpha_i(N) g(Y)^i.$$

Then, for $i < d$,

$$e_i = \frac{1}{2} \alpha_i(N).$$

Since $g(0) = 0$ and $g'(0) \neq 0$, we can apply the Bürmann-Lagrange theorem, in the version of [20, Lemma 8], to compute

$$\alpha_i(N) = \text{coeff. of } Y^i \text{ in } \frac{Y g'(Y)}{g(Y)} f(Y) \left(\frac{Y}{g(Y)} \right)^i =: \star$$

We compute

$$\star = (1 + Y)^{1-N+4i} (1 - Y)^{-2i-1} = (1 - Y^2)^{-2i-1} (1 + Y)^{2+6i-N}.$$

As $(1 - Y^2)^{-2i-1}$ is a power series in Y^2 with positive coefficients, we see that $\alpha_i(N)$ is positive if $2 + 6i - N > 0$, so if $i > \frac{N-2}{6}$. For $i < d$ we know that $\alpha_i(N) = 2e_i = (-1)^i 2^{-N+2+6i} \epsilon_i$ where ϵ_i is a non-negative integer, so $\alpha_i(N)$ is not positive for odd $i < d$.

Write $N = 12\mu + \rho$ with $0 \leq \rho \leq 7$ and assume that $d > 2\mu + 1$. Then $\alpha_{2\mu+1} > 0$ because $6(2\mu + 1) + 2 - (12\mu + \rho) = 8 - \rho > 0$ which is a contradiction. We conclude that $d \leq 2\mu + 1$ for $\rho = 0, 1, 2, 3, 5, 6, 7$. \square

We aim to find an analogous result to Proposition 4.1 for semi self-dual codes of length 24μ . So we need to find the bound $\text{dd}(\mathcal{D}) \leq 4\mu$ also for not doubly even semi-self dual codes \mathcal{D} of length 24μ . For certain values of μ , we may show that some coefficient of $F(x, y)$ is not integral.

Proposition 5.3. *If \mathcal{D} is not doubly-even and $n = 24\mu$ with $\binom{5\mu-1}{\mu-1}$ odd then $\text{d}(\mathcal{D}^\perp) \leq 4\mu$.*

Proof. With the notations used above, we get

$$\begin{aligned} \alpha_{2\mu}(12\mu) &= \text{coeff. of } Y^{2\mu} \text{ in } (1 - Y^2)^{-4\mu-1}(1 + 2Y + Y^2) \\ &= \text{coeff. of } Z^\mu \text{ in } (1 - Z)^{-4\mu-1} + \text{coeff. of } Z^{\mu-1} \text{ in } (1 - Z)^{-4\mu-1} \\ &= \binom{5\mu}{\mu} + \binom{5\mu-1}{\mu-1} = 6\binom{5\mu-1}{\mu-1}. \end{aligned}$$

On the other hand, assuming that $\text{d}(\mathcal{D}^\perp) \geq 4\mu + 2$, we have

$$\alpha_{2\mu}(12\mu) = 2e_{2\mu} = 2^2\epsilon_{2\mu}.$$

As $\epsilon_{2\mu}$ is a non-negative integer, we get that $\binom{5\mu-1}{\mu-1}$ is even. \square

It seems to be impossible to obtain the same bound for the other values of μ by just looking at weight enumerators. For $\mu = 5$ (the first value for which $\binom{5\mu-1}{\mu-1}$ is even), we get examples of $\{e_i\}$ for which $F(x, y)$ has non-negative integer coefficients and $B(1, y) = 1/2 + O(y^{22})$. From one of these we computed $W_{\mathcal{D}}(1, y) = 1 + O(y^{22})$, $W_{\mathcal{D}^\perp}(1, y) = 1 + O(y^{22})$ and $W_{S(\mathcal{D})}(1, y) = O(y^{18})$, all with non-negative integer coefficients.

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References

- [1] C. Aguilar Melchor, P. Gaborit, *On the classification of extremal $[36, 18, 8]$ binary self-dual codes*, IEEE Trans. Inform. Theory **54** (2008) 4743–4750.
- [2] C. Bachoc, *On harmonic weight enumerators of binary codes*, Des. Codes Cryptogr. **18** (1999) 11–28.
- [3] C. Bachoc, P. Gaborit, *Designs and self-dual codes with long shadows*, J. Combin. Theory A **105** (2004) 15–34.
- [4] M. Borello, *Automorphism groups of self-dual binary linear codes with a particular regard to the extremal case of length 72*, PhD Thesis, Università degli studi di Milano-Bicocca 2014.
- [5] M. Borello, W. Willems, *Automorphisms of order $2p$ in binary self-dual extremal codes of length a multiple of 24*, IEEE Trans. Inform. Theory **59** (2013) 3378–3383.
- [6] M. Borello, *The automorphism group of a self-dual $[72, 36, 16]$ binary code does not contain elements of order 6*, IEEE Transactions on Information Theory **58** (2012) 7240–7245.
- [7] M. Borello, *The automorphism group of a self-dual $[72, 36, 16]$ code is not an elementary abelian group of order 8*, Finite Fields and Their Applications **25** (2014) 1–7.
- [8] M. Borello, F. Dalla Volta and G. Nebe, *The automorphism group of a self-dual $[72, 36, 16]$ code does not contain S_3 , A_4 or D_8* , Advances in Mathematics of Communications **7** (2013) 503–510.
- [9] S. Bouyuklieva, *On the automorphisms of order 2 with fixed points for the extremal self-dual codes of length $24m$* , Des. Codes Cryptogr. **25** (2002) 5–13.
- [10] S. Bouyuklieva, J. de la Cruz and W. Willems, *On the automorphism group of a binary self-dual $[120, 60, 24]$ code*, Appl. Algebra Engrg. Comm. Comput. **24** (2013), 201–214.

- [11] S. Bouyuklieva, W. Willems, and N. Yankov, *On the Automorphisms of Order 15 for a Binary Self-Dual* [96, 48, 20] *Code*, arXiv:1403.4735 (2014).
- [12] L. Chouinard, *Projectivity and relative projectivity over group rings*, J. Pure and Applied Algebra **7** (1976) 278–302.
- [13] J.H. Conway and V. Pless, *On primes dividing the group order of a doubly-even (72; 36; 16) code and the group order of a quaternary (24; 12; 10) code*, Discrete Mathematics **38** (1982) 143–156.
- [14] P. Gaborit, *A bound for certain s-extremal lattices and codes*, Archiv der Mathematik **89** (2007) 143–151.
- [15] A. Günther and G. Nebe, *Automorphisms of doubly even self-dual binary codes*, Bulletin of the London Mathematical Society **41** (2009) 769–778.
- [16] C.L. Mallows and N.J.A. Sloane, *An upper bound for self-dual codes*, Inf. Control **22** (1973) 188–200.
- [17] A. Munemasa, *Database of binary self-dual codes*, Online available.
- [18] G. Nebe, *An extremal [72, 36, 16] binary code has no automorphism group containing $Z_2 \times Z_4$, Q_8 , or Z_{10}* , Finite Fields and their applications **18** (2012) 563–566.
- [19] E.M. Rains, *Bounds for Self-Dual Codes Over Z_4* , Finite Fields and Their Applications **6** (2000) 146–163.
- [20] E.M. Rains, *Shadow bounds for self-dual codes*, IEEE Trans. Inform. Theory **44** (1998) 134–139.
- [21] N.J.A. Sloane, *Is there a (72; 36) $d = 16$ self-dual code?*, IEEE Trans. Inform. Theory **2** (1973) 251.
- [22] V. Yorgov and D. Yorgov, *The Automorphism Group of a Self-Dual [72, 36, 16] Code Does Not Contain Z_4* , IEEE Trans. Inform. Theory **60** (2014) 3302–3307.
- [23] S. Zhang, *On the nonexistence of extremal self-dual codes*, Discrete Appl. Math. **91** (1999) 277–286.